Quantum chaotic trajectories in integrable right triangular billiards

J. A. de Sales*
Departamento de Física, Universidade Federal de Minas Gerais, 30.123-970 Belo Horizonte, Minas Gerais, Brazil

J. Florencio†
Instituto de Física, Universidade Federal Fluminense 24.210-340 Niterói, Rio de Janeiro, Brazil

(Received 14 October 2002; published 29 January 2003)

Right triangular billiards are very simple systems that are completely integrable in classical mechanics for acute angle pairs (45°,45°) and (30°,60°). In quantum mechanics, the energy level spacing distribution of these billiards are neither Poisson-like nor Wigner-like. We use Bohm's formalism to calculate the trajectories, by numerical methods, for a particle inside these billiards. We use a linear combination of the first three energy states as the initial wave function. We show that a particle can have quasiperiodic or chaotic behavior, depending on its initial position in the billiards.

DOI: 10.1103/PhysRevE.67.016216 PACS number(s): 05.45.−a, 03.65.Ge, 41.20.Jb

Chaos in quantum systems has received a lot of attention since it was conjectured that a signature of chaos could be provided by random matrix theory [1,2]. Level spacing statistics of quantum systems that are classically chaotic is supposed to yield a Gaussian-like distribution. This Wigner surmise was conjectured by Bohigas *et al.* [3]. Despite its success in many cases, there are a few examples in which such conjecture fails [4]. In fact, the signature of chaos in quantum systems is still a subject of controversy.

The current studies of quantum billiards in conventional quantum mechanics have shown behaviors that are similar to those in the classical world: integrable classical systems are also integrable in quantum mechanics; ergodic classical systems are also ergodic in quantum mechanics. Integrable systems yield a Poisson-like distribution for the neighboring level spacing statistics, while ergodic systems produce a Gaussian-like distribution. A complete theory for quantum systems with intermediate statistics is still lacking. In integrable classical systems, we can use the semiclassical theory to quantize the classical Hamiltonian, while for ergodic systems, the analysis rests mainly on the behavior of the Wigner function in the phase space. However, the quantization of classical Hamiltonians of systems that are neither integrable nor ergodic is not trivial.

The application of causal quantum mechanics theory, following Bohm's interpretation, in the study of billiards and in other systems have shown striking surprises when we analyze a single Bohmian trajectory of an individual system [5,6]. Some authors have found chaotic trajectories in systems that are integrable in the classical domain. Konkel and Makowski [7] studied the Bohmian trajectories of a particle in a square billiard and found evidence of chaotic behavior based on an analysis of the patterns of the nodal lines. The same system was investigated by Bonfim *et al.* [8], who found that a initial wave function consisting of energy eigenstates must necessarily have at least three components in order to obtain chaos. Chaos in Bohmian mechanics were also reported in nonisotropic harmonic oscillators [9] and in systems of noninteracting particles [10].

The present work aims at the study of Bohmian trajectories in triangular billiards. Our purpose is to determine the trajectories for several initial positions, however, with the same initial wave function. As we shall see below, we obtain both regular and chaotic solutions, as in classical systems.

In order to study the dynamics of quantum systems in causal Bohm's theory [11–13], one first takes the wave function $\psi(x,y,t)$ in polar form, i.e., $\psi = R\exp(-iS/\hbar)$, into Schrödinger equation. After some algebraic manipulations, that equation separates into two equations, one for the amplitude R and the other for the phase S, as follows:

$$\frac{\partial R^2}{\partial t} + \boldsymbol{\nabla}\left(\frac{R^2 \boldsymbol{\nabla} S}{M}\right) = 0, \tag{1}$$

$$\frac{\partial S}{\partial t} + \frac{(\boldsymbol{\nabla} S)^2}{2M} + V + Q = 0, \tag{2}$$

where V is the ordinary quantized potential and M is the mass of the particle. The quantity $Q = -(\hbar^2/2M)(\nabla^2 R/R)$ is the so-called quantum potential. Equation (1) represents the conservation of probability flux, whereas Eq. (2) can be identified as a generalized Hamilton-Jacobi equation with the usual potential replaced by the effective potential $V_{eff} = V + Q$.

In Bohm's theory, an additional postulate is introduced: the momentum of the particle is defined by

$$\mathbf{p} = \boldsymbol{\nabla} S. \tag{3}$$

Hence, the velocity of the particle, $\mathbf{v} = \mathbf{p}/M$, can be expressed in terms of the wave function

$$\mathbf{v}(x,y,t) = \frac{\hbar}{2Mi}(\psi^* \boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*)/(\psi^*\psi). \tag{4}$$

*Electronic address: jsales@fisica.ufmg.br
†Electronic address: jfj@if.uff.br

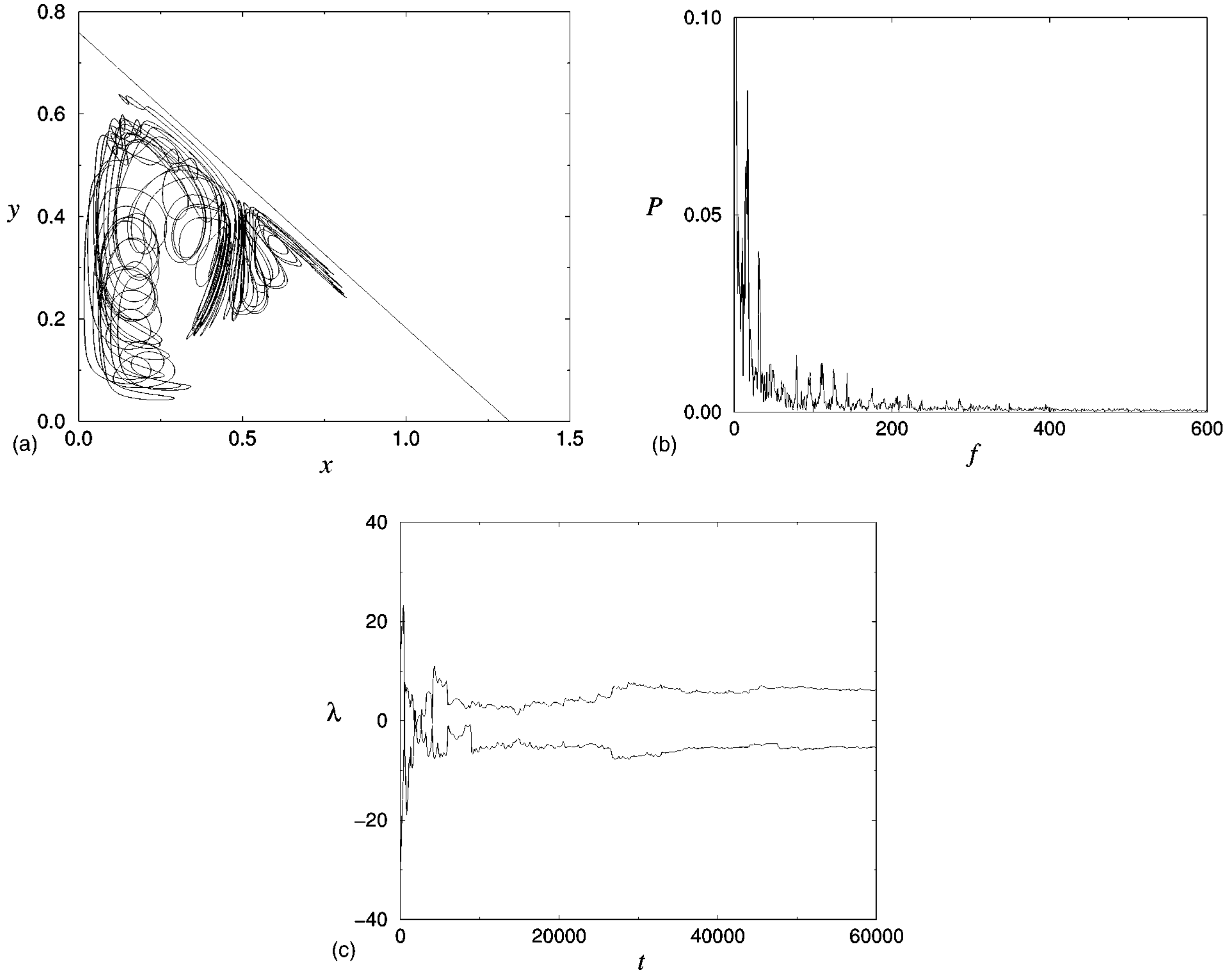


FIG. 1. Particle in a $(30°,60°)$-right triangular billiard in a chaotic regime. The initial wave function is given by Eq. (10), with initial position $(x_0=0.1,\ y_0=0.5)$. The system of units used here and in the following figures is such that $L=1$ and $\hbar^2/2ML^2=1$. From top to bottom: (a) Bohmian trajectory in the xy plane; (b) power spectral density as a function of frequency obtained from the power series of $x(t)$; (c) Lyapunov exponent λ as a function of time.

Note that the position and velocity of the particle are well defined at all times, provided the initial position does not lie on a node of the wave function.

It follows that the particle trajectory satisfies the modified Newton's second law:

$$M\frac{d^2\mathbf{r}}{dt^2}=-\boldsymbol{\nabla}(V+Q)\big|_{\mathbf{r}=\mathbf{r}(t)}. \tag{5}$$

In most cases of interest, this equation is hard to solve analytically, so one resorts to numerical methods. In this work, we integrate it numerically by using the fourth-order Runge-Kutta method [14], with the initial conditions $\mathbf{r}(t=0)=\mathbf{r}_0$ and $\mathbf{v}(t=0)=\mathbf{v}_0$, where $\mathbf{v}_0$ depends on the initial position of the particle in the billiards domain.

We studied the Bohm's trajectories in right triangle billiards for rational angles $\alpha=p/q\pi$, with $\alpha=\pi/4$ and $\alpha=\pi/6$ [15]. It is known that these billiards exhibit full integrability when in the classical regime, with trajectories lying on a bidimensional torus in the phase space. These two cases are the only ones which are completely integrable, i.e., one can always construct a bidimensional invariant surface embedded in a quadridimensional phase space. The billiards are pseudointegrable for other rational angles, and their trajectories wander throughout the whole phase space [16].

The wave functions satisfy Schrödinger equation with null boundary conditions at the billiard sides. In this work, we use scaling lengths and energies such that $L=1$ and $\hbar^2/2ML^2=1$. In the case of the $(30°,60°)$ billiard, the eigenfunctions and energies are given, respectively, by

$$\begin{aligned}\phi_{m,n}(x,y)=&\sin\left(\frac{\pi}{\sqrt[4]{3}}n(\sqrt[4]{3}-x)\right)\sin\left(\frac{\pi}{\sqrt[4]{27}}(n-2m)y\right)\\&+\sin\left(\frac{\pi}{\sqrt[4]{3}}m(\sqrt[4]{3}-x)\right)\sin\left(\frac{\pi}{\sqrt[4]{27}}(2n-m)y\right)\\&-\sin\left(\frac{\pi}{\sqrt[4]{3}}(n-m)(\sqrt[4]{3}-x)\right)\sin\left(\frac{\pi}{\sqrt[4]{27}}(n+m)y\right)\end{aligned} \tag{6}$$

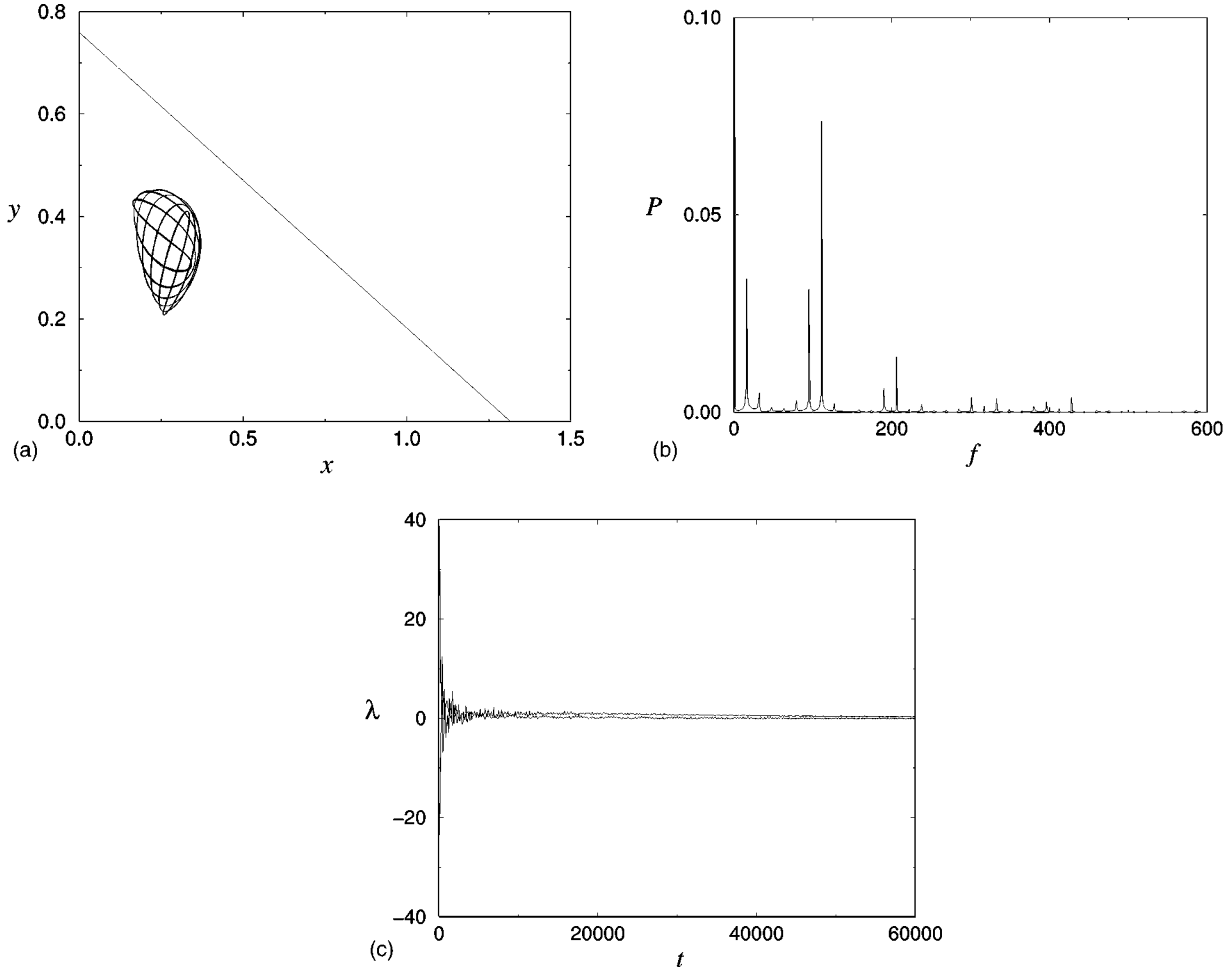


FIG. 2. Particle in a (30°,60°)-right triangular billiard in a quasiperiodic regime. The initial wave function is the same used in Fig. (1). The initial position is ($x_0=0.35$, $y_0=0.4$). From top to bottom: (a) real space Bohmian trajectory; (b) power spectrum from the time series of $x(t)$; (c) time evolution of the Lyapunov exponent.

and

$$E_{n,m}=\frac{4\pi^2}{\sqrt{[27}}(n^2+m^2-nm), \tag{7}$$

with the constraints $n\neq m,\ n\neq 2m,\ m\neq 2n$. In the case of the (45°,45°) billiard, the eigenfunctions and energies are given by

$$\psi_{m,n}(x,y)=\sin(\pi nx)\sin[\pi m(1-y)] - \sin(\pi mx)\sin[\pi n(1-y)] \tag{8}$$

and

$$E_{n,m}=\pi^2(n^2+m^2), \quad n\neq m. \tag{9}$$

We should point out that the level spacing statistics of these billiards does not produce a Poisson-like distribution $P(t)\sim e^{-t}$ as expected for such classically integrable systems [15].

To analyze the behavior of the Bohmian trajectories in those billiards, which are obtained from numerical integration of Eq. (4), we use well-established traditional methods from classical dynamics: Fourier spectra of the time series $x=x(t)$ and direct calculation of the time evolution of the Lyapunov exponents [17]. In all cases analyzed, we used the fourth-order Runge-Kutta method with a time-step integration of $h=10^{-4}$. The initial wave function was kept fixed, while we used several initial positions in the billiard. In this way, we found some situations, where the trajectory is associated to a chaotic regime and others to a quasiperiodic regime. These different kinds of dynamics are due to the strong variations of the time-dependent quantum potential in the billiards. One should also note that in classical dynamics, there are always periodic orbits amidst chaotic trajectories in the phase space.

Let us consider first the integrable (30°,60°) billiard. We choose the linear combination of the billiard energy eigenfunctions

$$\Psi(x,y,0)=\phi_{2,3}+\phi_{1,5}+i\phi_{3,5} \tag{10}$$

as the initial wave function, with $\phi_{m,n}$ given by Eq. (6). Note that the wave function is complex so as to ensure that the particle has a nonzero initial velocity [11]. As one instance of chaotic behavior, we present the results obtained for the ini-

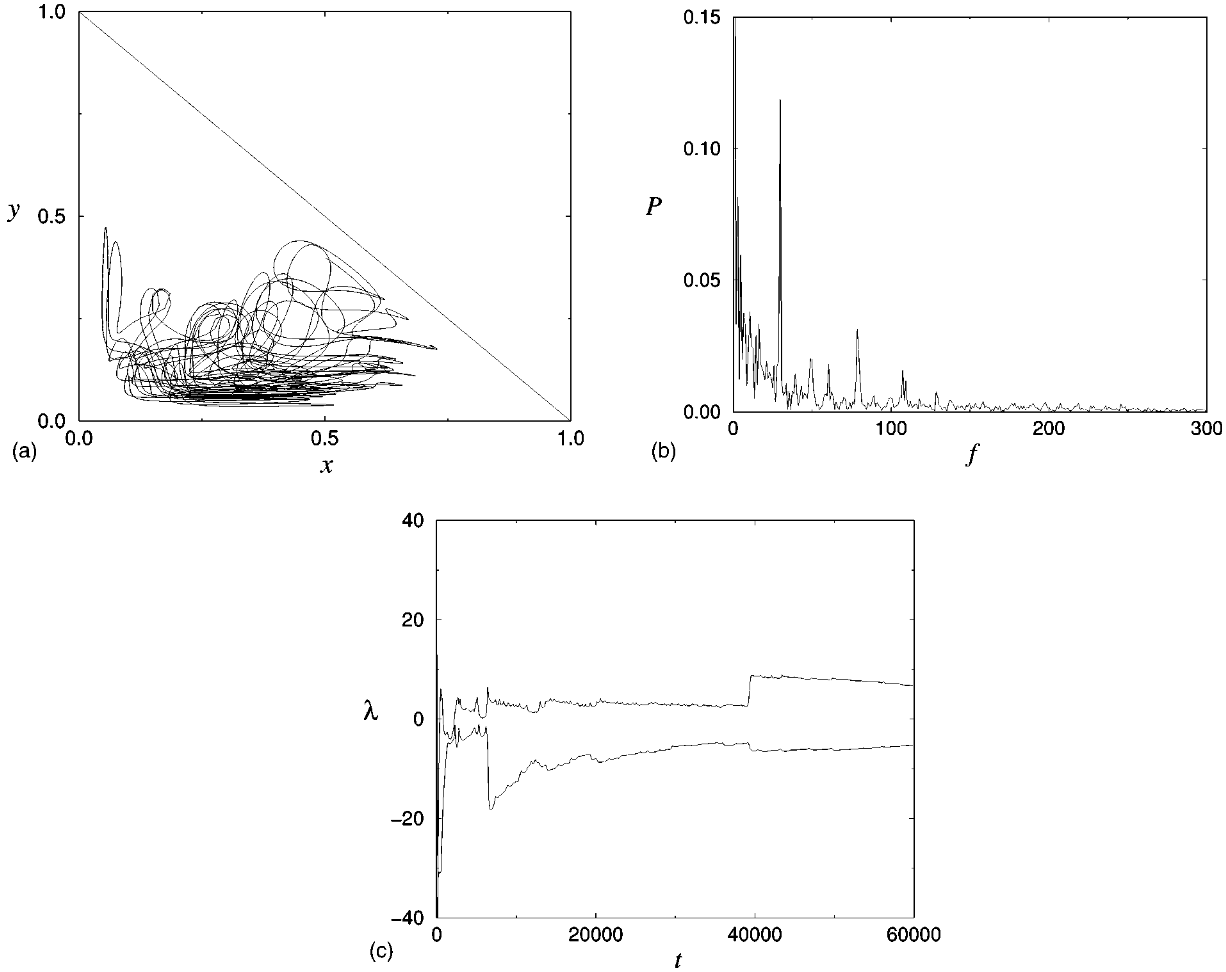


FIG. 3. Particle in a (45°,45°)-right triangular billiard in a chaotic regime. The initial wave function is given by Eq. (11), and the initial position is ($x_0=0.5$, $y_0=0.4$). From top to bottom: (a) Bohmian trajectory; (b) power spectrum of the time series $x(t)$; (c) Lyapunov exponent as a function of time.

tial position ($x_0=0.1$, $y_0=0.5$). Figure 1(a) shows the resulting chaotic Bohmian trajectory in the xy plane. We note that the trajectory visits a wide area of the billiard. However, it never hits the walls, where the quantum potential is strongly repulsive. The chaotic nature of the trajectory is also evident in the power spectrum of the time series of $x(t)$ depicted in Fig. 1(b). The broad band in the frequency spectrum is a typical signal of chaos. A quantitative measure of the nature of the trajectories is given by the Lyapunov exponent λ as a function of time. Our results for λ are shown in Fig. 1(c). The positiveness of the largest Lyapunov exponent in the figure indicates strongly that the trajectory in the phase space is very sensitive to small variations of the initial conditions. That is, the trajectory is chaotic.

In Fig. 2, we changed the initial position to ($x_0=0.35$, $y_0=0.4$) in the (30°,60°) billiard, but kept the same initial wave function, Eq. (10). In contrast to the previous case, we now obtain regular motion, with the quantum trajectory lying within a limited region, thus forming a caustic, as shown in Fig. 2(a). Therefore, the trajectory is not chaotic [18]. The power spectrum, shown in Fig. 2(b), has sharp peaks which depend on the basic frequencies of the system $\omega=(E_i-E_j)/\hbar$ and their harmonics, with i and j representing energy quantum numbers. Note also that the power spectrum does not show any background noise. These are features of a regular, quasiperiodic regime. The largest Lyapunov exponent, which is shown in Fig. 2(c), falls off to zero exponentially, as expected for systems in the regular regime. Such periodic motion is similar to the periodic orbits found in classical systems which exhibit chaos.

Similarly, for the (45°,45°) billiard, we can have both chaotic and regular motion, depending on the initial position. We consider the cases where the initial wave function is given by

$$\Psi(x,y,0)=\psi_{1,2}+\psi_{1,3}+i\,\psi_{2,3}, \tag{11}$$

with the components $\psi_{m,n}$ given by Eq. (8). First, let us analyze the results for the initial position ($x_0=0.5$, $y_0=0.4$). Figure 3(a) shows a chaotic trajectory in the xy plane. Again, we have a nonregular trajectory that spans over a large portion of the billiard, but it never hits the walls. The power spectrum, Fig. 3(b), displays the features associated to chaos, that is, broad background band with a few sharp

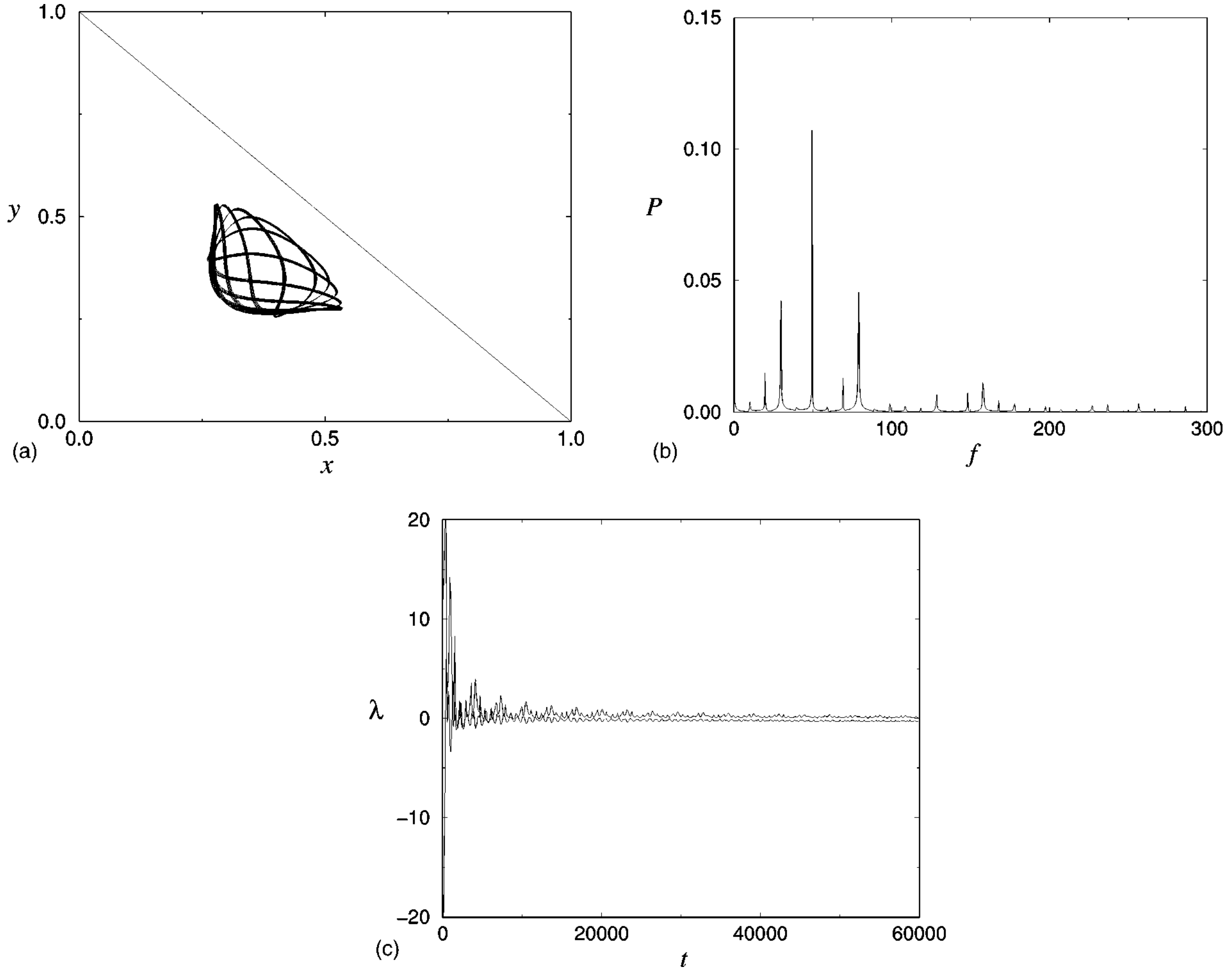
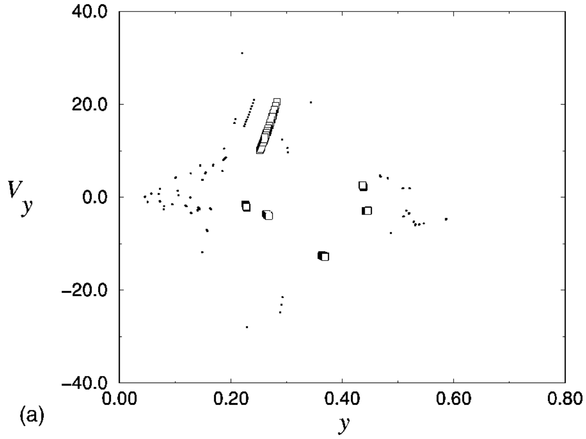


FIG. 4. Particle in a (45°,45°)-right triangular billiard in a quasiperiodic regime. The initial wave function is the same used in Fig. (3), and the initial position is ($x_0=0.3$, $y_0=0.5$). From top to bottom: (a) Bohmian trajectory; (b) power spectrum of the time series $x(t)$; (c) time evolution of the Lyapunov exponent.

peaks. Moreover, the largest Lyapunov exponent λ, seen in Fig. 3(c), is positive so that the trajectory thus obtained is definitely chaotic. By taking now a different initial position, ($x_0=0.3$, $y_0=0.5$), we obtain regular motion, with the trajectory concentrated in the central area of the billiard, resembling a Lissajous figure, which can be seen in Fig. 4(a). Such trajectory is quasiperiodic, however, it could also be inferred easily from the power spectrum, Fig. 4(b), and the from vanishing of the largest Lyapunov exponent at long times, as seen in Fig. 4(c).

It is worthwhile to take a look at the stroboscopic Poincaré sections corresponding to the trajectories discussed above, Figs. 5(a) and 5(b). The top plot is for the (30°,60°)—whereas the bottom plot is for the (45°,45°) billiard. Each plot shows an isolated set of points (represented here by squares) that lie on a single curve for the quasiperiodic trajectories, and a set of points corresponding to chaotic trajectories that span the entire phase space. That is, indeed, a clear indication of the difference between the two regimes. What is really surprising with Bohmian trajectories is the fact that one can still find stable periodic orbits in classically integrable triangles, since the quantum potential is usually nonlinear (as well as time dependent), which would likely foster chaotic behavior. Actually, for a given initial wave packet, our numerical data shows that only for some special initial positions did we find quasiperiodic behavior. We performed a large amount of calculations, using different wave packets, sweeping the billiards areas for initial positions at regular intervals. It just seems that almost all the initial positions yield chaos, except those drawn from a few regions of the phase space, in which the trajectories are stable quasiperiodic orbits. However, we did not find evidence for any discernible pattern of initial positions that lead to quasiperiodic behavior. Note that unstable periodic orbits are usually not attainable in numerical integration due to rounding-off errors, which drive the trajectory away from its due course, resulting in chaotic behavior. Since the location of regions containing initial positions that yield quasiperiodic orbits depends on the choice of the initial wave packet, we were unable to obtain any further insight on the dependence of the location of initial positions with the appearance of regular behavior.

To summarize, despite exhibiting full integrability in classical mechanics, the (45°,45°)- and (30°,60°)-right triangu-



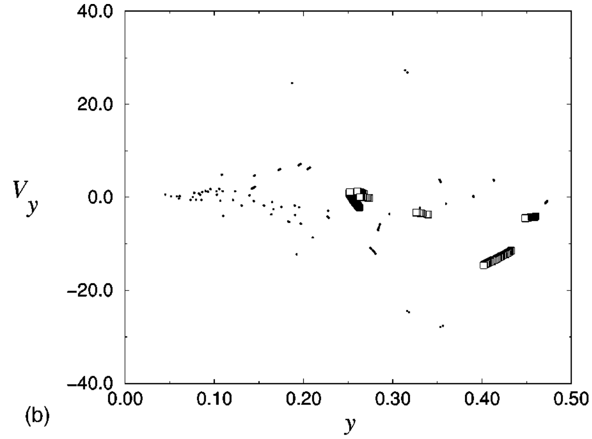


FIG. 5. Stroboscopic plots for the Bohmian trajectories shown in Figs. (1)–(4). Top (a), (30°,60°) billiard; bottom (b), (45°,45°) billiard. The squares are from quasiperiodic trajectories while the scattered points are from chaotic trajectories.

lar billiards in their quantum versions allow for chaos as well as for periodic behavior within Bohm's formalism. This depends on the initial position of the particle and the choice of initial wave packet. Thus, both regular and chaotic behaviors can also be found in other trajectories depending on the choice of initial positions with same initial wave packet.

We would like to thank O.F. de Alcantara Bonfim and F.C. Sá Barreto for fruitful discussions. One of us (J.F.) would like to thank the Departamento of Física da Universidade Federal de Minas Gerais, where most of this work was done. This work was partially supported by CNPq, FAPEMIG, and PRONEX (Brazilian agencies).

[1] T. Guhr, A. Müller-Groeling, and H.A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
[2] M.L. Mehta, *Random Matrix Theory* (Academic Press, New York, 1991).
[3] O. Bohigas, M.J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[4] J. Zakrzewski, K. Dupret, and D. Delande, Phys. Rev. Lett. **74**, 522 (1995).
[5] C. Dewdney and B.J. Hiley, Found. Phys. **12**, 27 (1982).
[6] C. Philippidis, C. Dewdney, and B.J. Hiley, Nuovo Cimento B **52B**, 15 (1979).
[7] S. Konkel and A.J. Makowski, Phys. Lett. A **238**, 95 (1998).
[8] O.F. de Alcantara Bonfim, J. Florencio, and F.C. Sá Barreto, Phys. Rev. E **58**, 2693 (1998).
[9] R.H. Parmenter and R.W. Valentine, Phys. Lett. A **201**, 201 (1995).
[10] R.H. Parmenter and R.W. Valentine, Phys. Lett. A **227**, 5 (1997).
[11] D. Bohm, Phys. Rev. **85**, 166 (1952); **85**, 180 (1952).
[12] D. Bohm and B.J. Hiley, *The Undivided Universe* (Routledge, London, 1993).
[13] P.R. Holland, *The Quantum Theory of Motion* (Cambridge University Press, Cambridge, 1993).
[14] W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, *Numerical Recipes in C* (Cambridge University Press, Cambridge, 1992).
[15] H.C. Schachner and G.M. Obemair, Z. Phys. B: Condens. Matter **95**, 113 (1994).
[16] P.J. Richens and M.V. Berry, Physica D **2**, 495 (1981).
[17] G. Rangarajan, S. Habib, and R.D. Ryne, Phys. Rev. Lett. **80**, 3747 (1998).
[18] R. Ramaswamy and R.A. Marcus, J. Chem. Phys. **74**, 1385 (1985).